SAGBI bases and Degeneration of Spherical Varieties to Toric Varieties

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Abstract. Let $X \subset \mathbb{P}(V)$ be a projective spherical G-variety, where V is a finite dimensional G-module and $G = \mathrm{SP}(2n, \mathbb{C})$. In this paper, we show that X can be deformed, by a flat deformation, to the toric variety corresponding to a convex polytope $\Delta(X)$. The polytope $\Delta(X)$ is the polytope fibred over the moment polytope of X with the Gelfand-Cetlin polytopes as fibres. We prove this by showing that if X is a horospherical variety, e.g. flag varieties and Grassmanians, the homogeneous coordinate ring of X can be embedded in a Laurent polynomial algebra and has a SAGBI basis with respect to a natural term order. Moreover, we show that the semi-group of initial terms, after a linear change of variables, is the semi-group of integral points in the cone over the polytope $\Delta(X)$. The results of this paper are true for other classical groups, provided that a result of A. Okounkov on the representation theory of $\mathrm{SP}(2n, \mathbb{C})$ is shown to hold for other classical groups.

Key words: SAGBI basis, horospherical variety, spherical variety, toric degeneration, Gelfand-Cetlin polytope, Newton polytope.
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Contents

1 Introduction

Let $X \subset \mathbb{P}(V)$ be a (normal) projective G-variety, where G is a classical group and V is a finite dimensional G-module. Suppose X is *spherical*, that is a Borel subgroup has a dense orbit. Generalizing the case of toric varieties, one can associate an integral convex polytope $\Delta(X)$ to X such that the Hilbert polynomial $h(t)$ of X is the Ehrhardt polynomial of $\Delta(X)$, i.e. $h(t) =$ number of integral points in $t\Delta(X)$. The polytope $\Delta(X)$ is the polytope fibred over the moment polytope of X with the Gelfand-Cetlin polytopes as fibres. This polytope was defined by A. Okounkov in [13], based on the results of M. Brion. We call this polytope the *Newton polytope* of X.

In this paper, for $G = \mathrm{SP}(2n, \mathbb{C})$, we show that X can be deformed (degenerated), by a flat deformation, to the toric variety corresponding to the polytope $\Delta(X)$ (Corollary 6.5). This is the consequence of the main result of the paper, i.e. the homogeneous coordinate ring of a *horospherical* variety has a SAGBI basis (Theorem 6.1). A spherical variety is horospherical if the stabilizer of a point in the dense G-orbit contains a maximal unipotent subgroup. Flag varieties and Grassmanians are examples of horospherical varieties. It is known that any spherical variety can be deformed, by a flat deformation, to a horospherical variety such that the moment polytopes of the two varieties are the same (see [14], [1, §2.2], [10, Satz 2.3]).

More precisely, we prove that if $X \subset \mathbb{P}(V)$ is a projective horospherical G-variety where $G = \mathrm{SP}(2n, \mathbb{C})$, the homogeneous coordinate ring R of X can be embedded in a Laurent polynomial algebra and has a SAGBI basis with respect to a natural term order [1]. Moreover, we show that the semi-group of initial terms is the semi-group of integral points in the cone over the polytope

[1]SAGBI stands for *Subalgebra Analogue of Gröbner Basis for Ideals.*

$\Delta(X)$. A finite collection $f_1, \dots, f_r$ of elements of R is a SAGBI basis, with respect to a term order, if the semi-group of initial terms is generated by the initial terms of the f_i and moreover, every element of R can be represented as a polynomial in the f_i, in a finite number of steps, by means of a simple classical algorithm called the *subduction algorithm*.

Degenerations of flag and Schubert varieties to toric varieties have been studied by Gonciulea and Lakshmibai in [9] and by Caldero in [5]. Recently, M. Kogan and E. Miller show the existence of a SAGBI basis for the coordinate ring of the flag variety of $\mathrm{GL}(n, \mathbb{C})$. More precisely, they prove that for any dominant weight λ in the interior of the Weyl chamber, the homogenous coordinate ring of the flag variety $\mathrm{GL}(n)/B$ embedded in $\mathbb{P}(V_\lambda)$ has a SAGBI basis and $\mathrm{GL}(n)/B$ can be degenerated to the toric variety corresponding to the Gelfand-Cetlin polytope of λ (see [11]). Main results of the present paper (Theorem 6.1 and Corollary 6.5) , in particular, imply the similar result for the flag varieties G/P of $G = \mathrm{SP}(2n, \mathbb{C})$.

A key step in our proof is a result of A. Okounkov on the representation theory of $\mathrm{SP}(2n, \mathbb{C})$. Let V_λ denote the irreducible G-module with highest weight λ, where $G = \mathrm{SP}(2n, \mathbb{C})$. It is well-known that one can view V_λ as a subspace of $\mathbb{C}[G]$ and, after restriction to U, as a subspace of $\mathbb{C}[U]$, where U is the standard maximal unipotent subgroup of G. In [12], Okounkov proves that, with respect to a natural term order on $\mathbb{C}[U]$, the set of highest terms of elements of V_λ can be identified with the Gelfand-Cetlin polytope Δ_λ (Theorem 5.2). As Okounkov informed the author, using similar methods used for $\mathrm{SP}(2n, \mathbb{C})$, one can prove his result for other classical groups. But so far he has not published the proofs for other classical groups. The results of the present paper as well as their proofs go verbatim for other classical groups, provided that Okounkov's result is shown to hold for them.

In Section 2, we discuss SAGBI bases. Section 3 deals with some facts about homogeneous coordinate ring of spherical varieties. We give a description of the homogeneous coordinate ring of a horospherical variety. In Section 4, we define the Gelfand-Cetlin polytopes and the polytope $\Delta(X)$. Section 5 discusses the result of A. Okounkov on the initial terms of elements of an irreducible G-module and Gelfand-Cetlin polytopes, for $G = \mathrm{SP}(2n, \mathbb{C})$. Finally, in Section 6 we state and prove our main results.

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2 SAGBI bases

In this section we define the notion of a SAGBI basis for a subalgebra of the Laurent polynomials. SAGBI bases play an important role when one deals with subalgebras of the polynomial or Laurent polynomial algebras. Their theory is more complicated than the theory of Gröbner bases. In particular, not every subalgebra has a SAGBI basis with respect to a given term order. It is an unsolved problem to determine, for a given term order, which subalgebras have a SAGBI basis.

Let $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ denote the algebra of Laurent polynomials in n variables. Let $\prec$ be a term order on $\mathbb{Z}^n$, that is a total order compatible with addition. An important example is the lexicographic order. The initial term, with respect to $\prec$, of a polynomial f is denoted by $\mathrm{in}(f)$. If R is a subalgebra of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, we denote by $\mathrm{in}(R)$ the semi-group of initial terms in R, i.e. $\{\mathrm{in}(f) \mid 0 \neq f \in R\}$.

First consider the case where R is a subalgebra of $\mathbb{C}[x_1, \ldots, x_n]$. In this case, one usually assumes that $\prec$ satisfies the extra condition:

$$\mathbf{a} \succ (0, \ldots, 0), \quad \forall \mathbf{a} \;\; 0 \neq \mathbf{a} \in \mathbb{N}^n.$$

Definition 2.1. Let R be a subalgebra of $\mathbb{C}[x_1, \ldots, x_n]$. A finite collection of polynomials $\{f_1, \ldots, f_r\} \subset R$ is a SAGBI basis for R, if $\{\mathrm{in}(f_1), \ldots, \mathrm{in}(f_r)\}$ generates the semi-group $\mathrm{in}(R)$.

When R has a SAGBI basis, one has a simple classical algorithm, due to Kapur-Madlener and Robbiano-Sweedler, to express elements of R in terms of the f_i as follows: Write $\mathrm{in}(f) = d_1 \mathrm{in}(f_1) + \cdots + d_r \mathrm{in}(f_r)$ for some $d_1, \ldots, d_r \in \mathbb{N}$. Dividing the leading coefficient of f by the leading coefficient of ${f_1}^{d_1} \cdots {f_r}^{d_r}$, we obtain a c such that the leading term of f is the same as the leading term of $c{f_1}^{d_1} \cdots {f_r}^{d_r}$. Set $g = f - c{f_1}^{d_1} \cdots {f_r}^{d_r}$. If $g = 0$, we are done; otherwise we replace f by g and proceed inductively. Since g has a smaller leading exponent than f, and $\mathbb{N}^n$ is well-ordered with respect to $\prec$, this process will terminate, resulting an expression for f as a polynomial in the f_i. This is referred to as *subduction algorithm*. See [16] for a detailed discussion of SAGBI bases for subalgebras of $\mathbb{C}[x_1, \ldots, x_n]$.

In general when R is a subalgebra of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, since $\mathbb{Z}^n$ is not well-ordered there is no guarantee that this algorithm terminates. Following [15, p. 2], we define the SAGBI basis as follows:

Definition 2.2. Let R be a subalgebra of $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. A finite collection of polynomials $\{f_1, \ldots, f_r\}$ is a SAGBI basis for R if:

(a) The $\mathrm{in}(f_i)$ generate $\mathrm{in}(R)$ as a semi-group; and

(b) the subduction algorithm described above terminates for every $f \in R$, no matter what choices are made for $d_1, \ldots, d_r$ in the course of the algorithm.

The algebra R is said to have a SAGBI basis, if it has a SAGBI basis for some choice of a term order.

3 Homogeneous coordinate ring of spherical and horospherical varieties

Let V be a finite dimensional G-module and $X \subset \mathbb{P}(V)$ a projective spherical G-variety, i.e. X is normal and a Borel subgroup $B \subset G$ has a dense orbit in X. Let $R = \mathbb{C}[X]$ denote the homogeneous coordinate ring of X. This algebra is graded by the degree of polynomials,

$$R = \bigoplus_{k=0}^{\infty} R_k.$$

We decompose the spaces R_k into irreducible G-modules,

$$R_k = \bigoplus_{\lambda} m_{\lambda,k} V_\lambda,$$

where V_λ is the irreducible G-module with the highest weight λ and $m_{k,\lambda}$ is its multiplicity. Since X is spherical its spectrum is multiplicity free, i.e. $m_{k,\lambda} \in \{0, 1\}$. Let $\Phi(X)$ denote the *moment polytope* of X, i.e. the intersection of the image of the moment map with the positive Weyl chamber for the choice of B. Also, denote by Λ the weight lattice of G. The following theorem due to Brion (see [3] and [4]) determines which weights λ occur in the decomposition of R_k with multiplicity 1:

Theorem 3.1 (Brion, §3 [4]). *There is a sublattice Λ' of Λ such that $\Phi(X) \subset \Lambda'_{\mathbb{R}}$, the vector space spanned by Λ', and we have:*

$$R_k = \bigoplus_{\lambda \in k\Phi(X) \cap \Lambda'} V_\lambda.$$

The rank of the sublattice Λ' is called the *rank* of the spherical variety X.

Remark 3.2. It follows from the above theorem that one can recover the moment polytope $\Phi(X)$ from the multiplicities of the irreducible G-modules appearing in R_k. More precisely, we have

$$\Phi(X) = \text{ closure of } \bigcup_{k=0}^{\infty} \{\frac{\mu}{k} \mid V_\mu \text{ appears in the decomposition of } R_k\}.$$

One can show that the ring multiplication in R sends $V_\lambda \times V_\mu$ to $V_{\lambda+\mu} \oplus \bigoplus_\nu V_\nu$, where $\nu = \lambda + \mu - \xi$ and ξ is some non-negative combination of simple roots. When all the stabilizer subgroups of the points of X contain a maximal unipotent subgroup, from a theorem of Popov (see [14, Theorem 2.3]) it follows that the ring multiplication sends $V_\lambda \times V_\mu$ to $V_{\lambda+\mu}$ and this map coincides with a Cartan multiplication. [2]

Definition 3.3. A spherical G-variety X such that the stabilizer of a point in the dense G-orbit contains a maximal unipotent subgroup is called a *horospherical* variety.

It can be shown that if X is horospherical, then all the stabilizer subgroups contain a maximal unipotent subgroup. Examples of horospherical varieties are toric varieties, flag varieties and Grassmanians.

Now, assume X is horospherical. Fix a point x in the dense G-orbit of X. Choose highest weight vectors f_λ in each simple submodule V_λ of R by the condition that $f_\lambda(x) = 1$. Then the product of these highest weight vectors is again such a vector, and for any two λ and μ appearing in the decomposition of R, one can uniquely define Cartan multiplication. We can then give the following description for the homogeneous coordinate ring of X:

Theorem 3.4. *We have the following isomorphism of graded algebras:*

$$R \cong \bigoplus_{k=0}^{\infty} \bigoplus_{\lambda \in k\Phi(X) \cap \Lambda'} V_\lambda,$$

[2]For definition of Cartan multiplication see [7, p. 429]

where the multiplication in the righthand side is defined as follows: Let $R_d = \bigoplus_\lambda V_\lambda$ and $R_e = \bigoplus_\mu V_\mu$ be the decomposition of two graded pieces of R. Then the multiplication $R_d \times R_e \to R_{d+e}$ is given by the Cartan multiplication $V_\lambda \times V_\mu \to V_{\lambda+\mu}$, defined uniquely by the above choice of the highest weight vectors f_λ and f_μ.

4 Newton polytope of a spherical variety

Let G be a classical group. In this section, following [13], we briefly explain the definition of the Newton polytope of a spherical G-variety X. We start by recalling Gelfand-Cetlin polytopes.

To each dominant weight λ of G, there corresponds a Gelfand-Cetlin (or briefly G-C) polytope Δ_λ. The convex polytope Δ_λ has the property that the number of integral points in Δ_λ is equal to the dimension of the irreducible G-module V_λ. The dimension of the Gelfand-Cetlin polytope is equal to the complex dimension of the maximal unipotent subgroup U of G, i.e. $\frac{1}{2}(\dim(G) - \mathrm{rank}(G))$. We recall the definition of Gelfand-Cetlin polytopes for $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{SP}(2n,\mathbb{C})$. For the definition of G-C polytopes for the orthogonal group see [2].

Definition 4.1 (G-C polytope for GL$(n,\mathbb{C})$). Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a decreasing sequence of integers representing a dominant weight in $\mathrm{GL}(n,\mathbb{C})$. The G-C polytope Δ_λ is the set of all real numbers $x_1, x_2, \ldots, x_{n-1}$, $y_1, \ldots, y_{n-2}, \ldots, z$, such that the following inequalities hold:

$$
\begin{array}{ccccccccccc}
\lambda_1 & & \lambda_2 & & \lambda_3 & \cdots & \lambda_{n-2} & & \lambda_{n-1} & & \lambda_n \\
& x_1 & & x_2 & & \cdots & & x_{n-2} & & x_{n-1} & \\
& & y_1 & & y_2 & \cdots & y_{n-3} & & y_{n-2} & & \\
& & & \cdots & & \cdots & & \cdots & & & \\
& & & & \cdots & & \cdots & & & & \\
& & & & & z & & & & &
\end{array}
$$

where the notation

$$
\begin{array}{ccc}
a & & b \\
& c &
\end{array}
$$

means $a \geq c \geq b$.

Definition 4.2 (G-C polytope for SP$(2n, \mathbb{C})$). Let B be the Borel subgroup of upper triangular matrices in SP$(2n, \mathbb{C})$ and the maximal torus of SP$(2n, \mathbb{C})$ be $\{(t_1, \ldots, t_n, {t_1}^{-1}, \ldots, {t_n}^{-1}) \mid t_i \in \mathbb{C}^*, \forall i = 1, \ldots, n\}$. Every dominant weight is then represented by a decreasing sequence of positive integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. The G-C polytope Δ_λ is the set of all real numbers $x_1, \ldots, x_n, y_1, \ldots, y_{n-1}, \ldots, z, w$, such that the following inequalities hold:

$$
\begin{array}{ccccccccc}
\lambda_1 & & \lambda_2 & & \ldots & \lambda_n & & 0 \\
& x_1 & & x_2 & \ldots & & x_n & \\
& & y_1 & & \ldots & y_{n-1} & & 0 \\
& & & \ldots & & & \ldots & \\
& & & & \ldots & & & \\
& & & & & z & & 0 \\
& & & & & & w &
\end{array}
$$

If the components of the weight λ are real, we still can define the Δ_λ by the above inequalities. So we can extend the definition of Δ_λ to all real λ.

Lemma 4.3. *The assignment $\lambda \mapsto \Delta_\lambda$ is linear, i.e. $\Delta_{c\lambda} = c\Delta_\lambda$ for any positive c and $\Delta_{\lambda+\mu} = \Delta_\lambda + \Delta_\mu$, where the addition in the righthand side is the Minkowski sum of convex polytopes.*

Proof. The proof is immediate from the definition in each of the three cases of classical groups. □

Now, let $X \subset \mathbb{P}(V)$ be a (smooth) projective spherical G-variety and $\Phi(X)$ its moment polytope. As before, let Λ denote the weight lattice and $\Lambda_\mathbb{R}$ the real vector space spanned by Λ.

Definition 4.4 (Newton polytope of a spherical variety). Define the set $\Delta(X) \subset \Lambda_\mathbb{R} \oplus \mathbb{R}^{\dim U} = \mathbb{R}^{\dim B}$, by

$$\Delta(X) = \bigcup_{\lambda \in \Phi(X)} (\lambda, \Delta_\lambda).$$

From Lemma 4.3, it follows that $\Delta(X)$ is a convex polytope.

Remark 4.5. In [13], as a corollary of a theorem of Brion, it is shown that the polytope $\Delta(X)$ has the property:

$$\dim R_k = \#\{k\Delta(X) \cap \Lambda'\},$$

where Λ' is the sublattice of Λ in Theorem 3.1. This means that the Hilbert polynomial of the variety X coincides with the Ehrhardt polynomial of the polytope $\Delta(X)$. Note that since the Hilbert polynomial of a toric variety corresponding to a polytope Δ is the Ehrhardt polynomial of Δ, and the Hilbert polynomial is invariant under a flat deformation, the above fact agrees with the main result of the paper, i.e. X can be deformed to the toric variety of the polytope $\Delta(X)$ (Corollary 6.5).

5 Initial terms of elements of an irreducible G-module and Gelfand-Cetlin polytopes

Let λ be a dominant weight and V_λ the corresponding irreducible G-module, where $G = \mathrm{SP}(2n, \mathbb{C})$. The purpose of this section is to explain the result of A. Okounkov in [12], regarding the initial terms of the elements of V_λ. We will need it in the proof of our main theorem.

First, we explain how one can identify V_λ with a subspace of a polynomial algebra, that is, the coordinate ring of the standard maximal unipotent subgroup. Let T be the standard maximal torus of diagonal matrices in G, B_+ the Borel subgroup of upper triangular matrices, and U_+ the maximal unipotent subgroup of B_+. Denote by B_- and U_- the opposite subgroups of B_+ and U_+ respectively. Fix a B_--eigenvector ξ in $(V_\lambda)^*$. It is well-known that the mapping from V_λ to $\mathbb{C}[G]$, defined by

$$v \mapsto f_v,$$

$$f_v(g) = \xi(g^{-1}v),$$

maps the G-module V_λ isomorphically to the subspace

$$\{f \in \mathbb{C}[G] \mid f(gb) = (-\lambda)(b)f(g), \forall b \in B_-\} \tag{1}$$

where $-\lambda$ is regarded as a character of B_-. We identify V_λ with its image in $\mathbb{C}[G]$. Choose the highest weight vector $v_\lambda \in V_\lambda$ such that $\xi(v_\lambda) = 1$.

Consider the Bruhat decomposition

$$G = \bigcup_{w \in W} B_+ w B_-,$$

where W is the Weyl group. We have $G/B_- = \bigcup_{w \in W} B_+ w B_- / B_-$ and, the big Bruhat cell $\mathcal{U}$ in G/B_- is B_+B_-. Since $B_+ \cap B_- = T$ and $B_+ = U_+T$, the

cell $\mathcal{U}$ can be identified with U_+, via $u \mapsto uB_-$. Since $\mathcal{U}$ is dense in G/B_-, every element of $V_\lambda \subset \mathbb{C}[G]$ is uniquely determined by its restriction to U_+. So we can consider V_λ as a subspace of $\mathbb{C}[U_+]$. Note that U_+ is isomorphic, as a variety, to the affine space of dimension $\frac{1}{2}(\dim(G) - \operatorname{rank}(G))$. One has:

Proposition 5.1. *The following diagram is commutative:*

$$
\begin{tikzcd}
V_\lambda \times V_\mu \arrow[r] \arrow[d, hook] & V_{\lambda+\mu} \arrow[d, hook] \\
\mathbb{C}[G] \times \mathbb{C}[G] \arrow[r] \arrow[d] & \mathbb{C}[G] \arrow[d] \\
\mathbb{C}[U_+] \times \mathbb{C}[U_+] \arrow[r] & \mathbb{C}[U_+]
\end{tikzcd}
$$

where the map in the first row is the Cartan multiplication, defined uniquely with the above choice of v_λ and v_μ, and the maps in the second and third rows are the usual product of functions.

Proof. From (1) it follows that each f_v defines a function on G/U_- and hence each V_λ can be identified with a subspace of $\mathbb{C}[G/U_-]$. Now the commutativity of the top part of the diagram follows from a theorem of Popov ([14, Theorem 2. 3], see also the paragraph after Remark 3.2). The commutativity of the bottom part of the diagram is trivial. $\square$

In [12], Okounkov interprets the G-C polytopes as the set of highest terms of the elements of the V_λ regarded as polynomials in $\mathbb{C}[U_+]$. Choose a basis $e_1, \ldots, e_{2n}$ of $\mathbb{C}^{2n}$ in which the matrix of the symplectic form is

$$
\begin{bmatrix}
 & & & & & 1 \\
 & 0 & & & \cdots & \\
 & & & 1 & & \\
 & & -1 & & & \\
 & \cdots & & & 0 & \\
-1 & & & & &
\end{bmatrix}.
$$

Let x_{ij} be the matrix elements in this basis. We use $x_{11}, \ldots, x_{nn}$ as coordinates in T and use the dual coordinates

$$g^\lambda = x_{11}^{\lambda_1} \cdots x_{nn}^{\lambda_n}, \quad g \in T, \lambda \in \Lambda,$$

for weights. The weights

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$$

are dominant for B_+.

We use $x_{ij}, i<j, i+j \leq 2n+1$, as coordinates in U_+, and the big Bruhat cell $\mathcal{U}$. Consider the following lexicographic ordering on $\mathbb{C}[U_+]$:

$$\prod x_{ij}^{p_{ij}} \succ \prod x_{ij}^{q_{ij}}$$

if $p_{1,2n} < q_{1,2n}$, or if $p_{1,2n} = q_{1,2n}$ and $p_{1,2n-1} < q_{1,2n-1}$, and so on. Note that in particular

$$x_{1,2n} \prec x_{1,2n-1} \prec \cdots \prec x_{12} \prec x_{2,2n-1} \prec \cdots \prec x_{23} \prec \cdots \prec x_{n,n+1}, \tag{2}$$

which is exactly the reverse of the ordering of positive roots induced by the standard lexicographic order in $\mathbb{R}^n$. For a dominant weight λ and a monomial

$$\prod x_{ij}^{p_{ij}},$$

put

$$\begin{array}{rcl} \eta_i & = & \lambda_i - p_{1,2n-i+1}, \quad i=1,\ldots,n, \\ \theta_i & = & \eta_{i+1} + p_{1,i+1}, \quad i=1,\ldots,n-1, \\ \eta'_i & = & \theta_i - p_{2,2n-i}, \quad i=1,\ldots,n-1, \\ \theta'_i & = & \eta'_{i+1} + p_{2,i+1}, \quad i=1,\ldots,n-2, \end{array} \tag{3}$$

Theorem 5.2 ([12], Theorem 2). *View V_λ as a subspace of $\mathbb{C}[U_+]$. Then, with the above grading on $\mathbb{C}[U_+]$, the monomial*

$$\prod x_{ij}^{p_{ij}}$$

is a highest monomial of a polynomial in V_λ if and only if the numbers $\eta_1,\ldots,\eta_n,\theta_1,\ldots,\theta_{n-1},\eta'_1,\ldots,\eta'_{n-1},\ldots$, belong to the G-C polytope Δ_λ.

Let us denote the vector $(\eta,\theta,\eta',\theta',\ldots) \in \mathbb{R}^{\dim U}$ by $(q_{ij}),\ i<j, i+j \leq 2n+1$. The change of variables $p_{ij} \mapsto q_{ij}$ in (3), can be written in the matrix form as:

$$(q_{ij}) = A(p_{ij}) + B\lambda, \tag{4}$$

where A is a constant upper triangular matrix with $0, 1$ and -1 as entries and $1, -1$ on the diagonal, and B is the matrix of the linear transformation

$$\lambda = (\lambda_1, \ldots, \lambda_n) \mapsto (\lambda_1, \lambda_2 \ldots, \lambda_n, \lambda_2, \lambda_3 \ldots, \lambda_n, \ldots, \lambda_n) \in \mathbb{R}^{\dim(U)}.$$

Note that $\det(A) = \pm 1$ and hence the inverse of A also has integer entries. From (4) we can write

$$(p_{ij}) = A^{-1}((q_{ij}) - B\lambda),$$

Now, Theorem 5.2 can be stated as follows: the monomial

$$\prod x_{ij}^{p_{ij}}$$

is a highest term of an element of V_λ if and only if $(p_{ij}) \in A^{-1}(\Delta_\lambda - B\lambda)$.

Definition 5.3. We denote the polytope $A^{-1}(\Delta_\lambda - B\lambda)$ by Δ'_λ.

One has $\Delta_\lambda = A\Delta'_\lambda + B\lambda$, and hence the two polytopes can be transformed to each other by integral translations and integral transformations. Thus Δ_λ and Δ'_λ are integrally equivalent. The following is immediate from the definition:

Lemma 5.4. *The map $\lambda \mapsto \Delta'_\lambda$ is linear, i.e. $\Delta'_{c\lambda} = c\Delta'_\lambda$ for a positive c, and $\Delta'_{\lambda+\mu} = \Delta'_\lambda + \Delta'_\mu$ where the addition in the righthand side is the Minkowski sum.*

Definition 5.5. For a spherical variety X, similar to the definition of $\Delta(X)$, define $\Delta'(X) \subset \Lambda_\mathbb{R} \oplus \mathbb{R}^{\dim U} = \mathbb{R}^{\dim B}$, by

$$\Delta'(X) = \bigcup_{\lambda \in \Phi(X)} (\lambda, \Delta'_\lambda).$$

From the above lemma, $\Delta'(X)$ is a convex polytope.

Remark 5.6. The map $(\lambda, x) \mapsto (\lambda, A^{-1}(x - B\lambda))$, is an integral transformation that maps $\Delta(X)$ to $\Delta'(X)$. The inverse of this transformation is $(\lambda, x) \mapsto (\lambda, Ax + B\lambda)$ which is also integral. So the polytopes $\Delta'(X)$ and $\Delta(X)$ can be transformed to each other by integral transformations and hence are integrally equivalent.

6 Main Theorem

In this section, we prove the main results of the paper.

Theorem 6.1. *Let V be a finite dimensional G-module, and $X \subset \mathbb{P}(V)$ a projective horospherical G-variety, where $G = \mathrm{SP}(2n, \mathbb{C})$. We have:*

(i) The homogeneous coordinate ring R of X can be embedded into the Laurent polynomial algebra $\mathbb{C}[x_1, \ldots, x_d, y_1^{\pm 1}, \ldots, y_r^{\pm 1}, t]$, where $d = \frac{1}{2}(\dim(G) - \mathrm{rank}(G))$ and $r = \mathrm{rank}(X)$.

(ii) R has a SAGBI basis with respect to a natural term order. Moreover, the semi-group of initial terms $S = \mathrm{in}(R) \subset \mathbb{Z}^{d+r+1}$ coincides with the semi-group of integral points in the cone over the polytope $\Delta'(X)$ (see Definitions 5.3 and 5.5), i.e.

$$S = \mathbb{Z}^{d+r+1} \cap \bigcup_{k=0}^{\infty} (k\Delta'(X), k).$$

Proof. We identify $\mathbb{C}[U_+]$ with the polynomial algebra $\mathbb{C}[x_1, \ldots, x_d]$ equipped with the term order $\prec$ in Theorem 5.2. For each λ, let ϕ_λ denote the embedding $V_\lambda \hookrightarrow \mathbb{C}[x_1, \ldots, x_d]$. Let Λ' be the sublattice of the weight lattice in Theorem 3.1. Let $C \cong (\mathbb{C}^*)^r$ be a torus whose lattice of characters is Λ'. Let $y_1, \ldots, y_r$ be a choice coordinates in C, hence $\mathbb{C}[C] = \mathbb{C}[{y_1}^{\pm 1}, \ldots, {y_r}^{\pm 1}]$. For $\lambda = (\lambda_1, \ldots, \lambda_r) \in \Lambda'$, and $y = (y_1, \ldots, y_r) \in C$, define $y^\lambda = {y_1}^{\lambda_1} {y_2}^{\lambda_2} \ldots {y_r}^{\lambda_r}$. Having the algebra isomorphism in Theorem 3.4 in mind, define the function

$$\Psi : R = \bigoplus_{k=0}^{\infty} \bigoplus_{\lambda \in k\Phi(X) \cap \Lambda'} V_\lambda \to \mathbb{C}[x_1, \ldots, x_d, {y_1}^{\pm 1}, \ldots, {y_r}^{\pm 1}, t],$$

by

$$\Psi(f) = t^k y^\lambda \phi_\lambda(f), \quad \forall f \in V_\lambda, \lambda \in k\Phi(X) \cap \Lambda'$$

where t is an extra free variable. Then we have

Lemma 6.2. Ψ *is an injective homomorphism of algebras.*

Proof. Since the ϕ_λ are additive homomorphisms, it follows that Ψ is also additive. The multiplicativity of Ψ follows from Proposition 5.1. Ψ is 1-1, because the ϕ_λ are 1-1. □

Now, R can be thought of as a subalgebra of $\mathbb{C}[x_1, \ldots, x_d, {y_1}^{\pm 1}, \ldots, {y_r}^{\pm 1}, t]$. Extend the term order $\prec$ to $\mathbb{C}[x_1, \ldots, x_d, {y_1}^{\pm 1}, \ldots, {y_r}^{\pm 1}, t]$ by lexicographic order such that $t \succ y_r \succ \cdots \succ y_1 \succ x_i,\ i = 1, \ldots, d$. Let $S = \mathrm{in}(R) \subset \mathbb{Z}^{d+r+1}$. From Theorem 5.2, we have

$$S = \mathbb{Z}^{d+r+1} \cap \bigcup_{k=0}^{\infty} \bigcup_{\lambda \in k\Phi(X) \cap \Lambda'} (\Delta'_\lambda, \lambda, k),$$

i.e. S is the semi-group of integral points in the cone over the polytope $\Delta'(X)$. This cone is a (strictly) convex rational polyhedral cone and hence S is finitely generated (Gordon's lemma). Also, from the definition of $\prec$ and S, there are only finitely many points in S which are smaller than a given point in S. This means that the subduction algorithm terminates after a finite number of steps. Thus R has a SAGBI basis and the proof of the theorem is finished. $\square$

Suppose R is an arbitrary subalgebra of a Laurent polynomial algebra. It is standard that the polynomials in R can be continuously deformed to their initial terms. More precisely, one can show that there is a flat family of algebras $\pi : \mathcal{R} \to \mathbb{C}$, such hat $\pi^{-1}(t) \cong R, \forall t \neq 0$ and $\pi^{-1}(0) = \mathbb{C}[\mathrm{in}(R)]$, the semi-group algebra of $\mathrm{in}(R)$ (see [6, Theorem 15.17]). If the semi-group $\mathrm{in}(R)$ is finitely generated then $\mathbb{C}[\mathrm{in}(R)]$ is the coordinate ring of an affine (possibly non-normal) toric variety. Geometrically speaking, this means that $\mathrm{Spec}(R)$ can be deformed, by a flat deformation, to this affine toric variety.

Corollary 6.3. *Let $G = \mathrm{SP}(2n, \mathbb{C})$. Any projective horospherical G-variety $X \subset \mathbb{P}(V)$ can be deformed, by a flat deformation, to the toric variety corresponding to the polytope $\Delta(X)$. That is, there exists a flat family of varieties $\pi : \mathcal{X} \to \mathbb{C}$, such that $\pi^{-1}(t) \cong X, \forall t \neq 0$ and $\pi^{-1}(0)$ is the toric variety of the polytope $\Delta(X)$.*

Proof. Let R be the homogeneous coordinate ring of X. From [6, Theorem 15.17, p. 343], we know that $\mathrm{Spec}(R)$ can be deformed, by a flat deformation, to the affine toric variety whose coordinate ring is the semi-group algebra $\mathbb{C}[S]$. Since $\Delta'(X)$ and $\Delta(X)$ can be transformed to each other by integral transformations (Remark 5.6), the semi-group S is isomorphic to S_0, the semi-group of integral points in the cone over $\Delta(X)$. So $\mathrm{Spec}(R)$ can be deformed to the toric variety $\mathrm{Spec}(\mathbb{C}[S_0])$. It is well-known that the projectivization of this affine toric variety is the toric variety corresponding to the polytope $\Delta(X)$ (see [17], p. 36). This finishes the proof of the corollary. $\square$

Now, let $X \subset \mathbb{P}(V)$ be a projective spherical G-variety. By a general result of Popov applied to the spherical varieties, one can deform X, by a flat deformation, to a horospherical variety X_0. More precisely:

Theorem 6.4 (see [14]; [1] §2.2; [10] Satz 2.3). *Let G be a reductive group and Y an affine spherical G-variety. There exists a flat family of affine G-varieties $\pi : \mathcal{Y} \to \mathbb{C}$ such that:*

1. *the $Y_t = \pi^{-1}(t)$ are isomorphic to Y as G-varieties for $t \neq 0$.*
2. *$Y_0 = \pi^{-1}(0)$ is horospherical.*
3. *$\mathbb{C}[Y]$ and $\mathbb{C}[Y_0]$ are isomorphic as graded G-modules, in particular the multiplicities of the irreducible representations V_λ appearing in the graded pieces $\mathbb{C}[Y]_d$ and $\mathbb{C}[Y_0]_d$ are the same, for any $d \geq 0$*

If $X \subset \mathbb{P}(V)$ is a projective spherical variety, let Y in the above theorem be the cone over X in V. We obtain that X can be degenerated to a projective horospherical variety X_0 where X_0 is the projectivization of Y_0 in the theorem. Since the multiplicities of the irreducible G-modules apparing in the homogenuous coordinate rings of X and X_0 are the same we see that the moment polytopes of X and X_0 are the same (see Remark 3.2). It is then immediate from the definition that $\Delta(X) = \Delta(X_0)$.

Corollary 6.5. *Let $G = \mathrm{SP}(2n, \mathbb{C})$. Any projective spherical G-variety $X \subset \mathbb{P}(V)$ can be deformed, by a flat deformation, to the toric variety corresponding to the polytope $\Delta(X)$. That is, there exists a flat family of varieties $\pi : \mathcal{X} \to \mathbb{C}$, such that $\pi^{-1}(t) \cong X, \forall t \neq 0$ and $\pi^{-1}(0)$ is the toric variety of the polytope $\Delta(X)$.*

Proof. By the above comment X can be deformed to a horospherical variety X_0 and $\Delta(X) = \Delta(X_0)$. The corollary now follows from Corollary 6.3. □

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